"On the Mysteries of Numbers alluded to by Fermat." By the Rt. Hon. Sir FREDERICK POLLOCK, Lord Chief Baron, F.R.S., &c.*

(Abstract.)

This paper is presented as a continuation of one which appeared in the Philosophical Transactions of the Royal Society for 1861, vol. cli. p. 409, and the object of it is to call attention to certain properties of odd numbers when placed in a square, according to an arrangement to be explained below.

It appears to me probable that these properties are connected with (if, indeed, they be not actually some form of) the mysterious properties of numbers, to which Fermat alludes in the announcement of his theorem (as furnishing the proof of it); for in point of fact these properties give *a method* by which every odd number can be divided into four square numbers, and every number (odd or even) can be divided into not exceeding three triangular numbers.

I am not prepared to say whether or not the method affords a *demonstration*, and proves that it *can* always be done; but it always *does* it, and the cause of its success may be distinctly shown. The properties I allude to are scarcely less interesting and curious than the theorem itself, and present results for which I can find no name more appropriate than the *geometry of numbers*, for relations appear to be established between various numbers in the square, which relations are not founded on any arithmetical connexion between them, but on the positions they respectively occupy in the square of which they form a part.

The arrangement of the numbers is as shown in diagram No. 1. Any odd number (which may be the subject of inquiry) is made the first term of a series, increasing from left to right by the numbers 4, 8, 12, 16, . . . $(4n)$ This series forms the horizontal line at the top; each term of the series so formed is made the first term of a series, increasing downwards by the numbers 2, 6, 10, 14 . . . . $(4n-2)$. A square of indefinite magnitude is thus formed, consisting of two sets of series, one set all horizontal, the other set all vertical. 161 is the first number in diagram No. 1.

The result of the arrangement of the two series in the manner above mentioned is the formation of a third set of series, which may be found in the diagonal lines of the square.

If from the first number in the square (161) a diagonal line be drawn towards the opposite corner, it will pass through the first terms of one portion of the third set of series; the second, third, and following terms are taken alternately from each side of the diagonal line. These series increase

* Read April 19. (See p. 106.)

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by 2, 4, 6, 8, &c. . . . ($2n$), and with the exception of one term they are all in the diagonal lines (see diagram No. 1), in which the single red line passes through the first terms, and the double red line shows where the terms of the series (the second, third, &c.) belonging to *that* (as a first term) are to be found; the red ink numbers indicate their order.

1 2 3 4 5 6 7 8

The Nos. 203, 205, 209, 215, 223, 233, 245, 259, &c., &c., compose the series; the terms increase by 2, 4, 6, 8, 10, &c. . . . $2n$. Any number in the diagonal from 161 may be the first term of a similar series.

The diagonal from 163 will give all the other first terms of the third set of series.

In order to explain the indices which appear in the diagram No. 1, and to show in what manner the series are connected with, and pass into each other, it is necessary to point out the properties of the two series which compose the square, and of the third series, which is a necessary result.

All of them have this property in common,—that if you can discover the roots of the squares which compose any term of the series with reference to the nature of the series, and the order in the series of that term, then you know the roots of every term in the series, both before and after that term. The first series increases by 4, 8, 12, &c. The indices of the terms of a series so increasing I have made 1, 3, 5, 7, &c., and they are put at the top as being common to all the series that are horizontal.

For this reason, if two numbers differ by 1, as n, $n+1$, and the larger be increased by 1, and the smaller be diminished by 1, and the process be continued, the result will be

$$\begin{array}{ll} n, & n+1, \\ n-1, & n+2, \\ n-2, & n+3, \\ n-3, & n+4, \\ \cdot & \cdot \\ \text{\&c.} & \text{\&c.} \\ n-(p-1) & n+p. \end{array}$$

If these be treated as roots, the sums of the squares will be

$$\begin{array}{l} 2n^2+2n+1, \\ 2n^2+2n+5, \\ 2n^2+2n+13, \\ 2n^2+2n+25, \\ \text{\&c.} \quad \text{\&c.} \\ 2n^2+2n+2p^2-2p+1. \end{array}$$

The sums of the squares increase by 4, 8, 12, &c.

If therefore the roots of the square, into which any odd number may be divided, be p, q, n, $n+1$, and the number be increased by 4, 8, 12, &c. ... $(4n)$, the mth term in the series will be composed of squares whose roots will be p, q, $n-(m-1)$, $n+m$; two of the roots will be constant, the others will vary, and their differences in the successive terms will be 1, 3, 5, 7, &c. $(2n-1)$; and if you discover the roots of any term in the series, you can find the roots of all the terms.

The second series resembles the first in having two roots constant, and two variable; the differences between the variable roots are, in the first term 0, in the second term 2, in the third term 4, &c., and the indices of the terms are therefore 0, 2, 4, 6, 8, 10, &c., which are placed vertically by the side of the square. For if two numbers are equal, as n, n, and one of them be increased by 1, and the other diminished by 1, and the process be continued, the result will be

n	n		
$n-1$,	$n+1$	If these be treated as	$2n^2$
$n-2$,	$n+2$	roots, the sums of	$2n^2+2$
$n-3$,	$n+3$	their squares will be	$2n^2+8$
&c.	&c.		$2n^2+18$
			&c. &c.

The sums of the squares increase by 2, 6, 10, 14 .... $(4n-2)$, and the successive differences of the variable roots are 0, 2, 4, 6, &c.; and if the roots of the squares into which any odd number may be divided be p, q, n, n, and the number be increased by 2, 6, 10, &c., the roots of the mth term will be p, q, $n-(m-1)$, $n+(m-1)$, and if the roots of any one term be known, the roots of all the others may be found.

The small figures in the upper right-hand corner of each division or small square are the indices of the third set of series. In this set all the roots are variable. The character of the first set of series is, that two roots in every term differ by an odd number; the character of the second set of series is, that two roots in every term differ by an even number; but in the third set of series, the algebraic sum of all the roots of the squares into which the successive terms may be divided is successively 1, 3, 5, 7, 9, &c. (an odd number): the sum of the roots of the squares into which an odd number can be divided cannot be an even number.

The following Table will explain in what manner the series is formed from the roots of the squares into which any odd number may be divided, so as to make the algebraic sum equal to 1. I have preferred to use figures instead of algebraic symbols, as being more readily understood and more easily dealt with; but the result is the same whatever figures or symbols may be used. The series begins from the centre.

Let $-7, -3, 2, 9$, which are the roots of the squares into which the odd number 143 may be divided, be placed in the centre, and let the positive roots be increased downwards and decreased upwards, and the nega-

tive roots increased upwards and decreased downwards, the result will be as in the Table below :—

	Order of of terms.	Roots.	Algebraic sums of roots.	Sums of squares of roots.	Order of terms.
		4 5 7			
	12	−13−9−4 3	−23	275	12
		4 5 7			
	10	−12−8−3 4	−19	233	10
		4 5 7			
	8	−11−7−2 5	−15	199	8
		4 5 7			
	6	−10−6−1 6	−11	173	6
		4 5 7			
	4	− 9−5 0 7	− 7	155	4
		4 5 7			
	2	− 8−4 1 8	− 3	145	2
		4 5 7			
Centre	1	− 7−3 2 9	1	143	1
		4 5 7			
	3	− 6−2 3 10	5	149	3
		4 5 7			
	5	− 5−1 4 11	9	163	5
		4 5 7			
	7	− 4 0 5 12	13	185	7
		4 5 7			
	9	− 3 1 6 13	17	215	9
		4 5 7			
	11	− 2 2 7 14	21	253	11
		4 5 7			
	13	− 1−3 8 15	25	299	13
		&c. &c.	&c.		

It will be observed that the terms of the series 143, 145, 149, 155, &c. increase by 2, 4, 6, 8, . . (2 *n*). In the column of the sums of the roots 1, 5, 9, 13, &c. increase by 4.

1, −3, −7, −11 decrease by 4; the differences of the roots, if arranged in the order of their numerical value, is always the same throughout the series.

Note.—In the remainder of this paper every odd number that becomes a term in any of the series is expressed by the roots the sum of whose squares form the number itself; | 2, 3, 6, 8 | means that the number occupying that division or small square is $113=(2^2+3^2+6^2+8^2)$; a figure (or collection of figures) representing merely the arithmetical value is put into a circle, thus: | (113) 2, 3, 6, 8 |.

Having explained the construction of the square and the indices which belong to the series of which it is composed, I propose to point out the properties which discover the roots of the squares into which the odd numbers (which are found in the different parts of the square) may be divided.

If the first odd number in the square be of the form $(4p+2).n+1$ (or $2n+1$, $6n+1$, $10n+1$, $14n+1$, &c.), n being of any value whatever, the $(n-p)$th term will be $\boxed{-(p+1), -p, n, n}$; these are the roots (the number itself would be $2n^2+2p^2+2p+1$); and as the index of the nth term is $n+(n-1)$, or $2n-1$, the index of the $(n-p)$th will be $2n-(2p+1)$, and the algebraic sum of the roots may be made equal to the index; it is therefore a term in a diagonal series [that the $(n-p)$th term is $2n^2+2p^2+2p+1$, will appear by finding in the usual way the $(n-p)$th term of a series whose first term is $(4p+2).n+1$, and the terms of which increase by 4, 8, 12, 16, &c. . . . $4n$]; but as two of the roots are equal, n, n, it is the first term of a vertical series descending, thus:

$p+1, p, n, n$
$p+1, p, (n-1), n+1$
$p+1, p, (n-2), n+2$
$p+1, p, (n-3), (n+3)$.

I call this term (A)* and indicate it by that letter, and from this term the roots of many others may be derived (which are indicated by other letters connected with A in an invariable manner), whose squares will compose the number that belongs to that term. For example, counting $2p+1$ squares backwards from A is a term I have distinguished as W; the roots of the number belonging to it are

$$\boxed{\overline{p+1}, p, (n-\overline{4p+2}), n}$$

These roots may, of course, be either positive or negative; arranging them thus, $-\overline{p+1}$, $-p$, n, $-\overline{4p+2}$, n, we have the algebraic sum of their roots equal to $2n-\overline{6p+3}$, which is the index of the square in which W is found going backwards from A, the index of which is $2n-\overline{2p+1}$ (as already stated); immediately adjoining W are two squares which I have called M and N respectively.

M is the term next before W, and is composed of the roots

$$\boxed{n-\overline{2p+2}, n-\overline{2p+2}, \overline{3p+2}, \overline{p+1}}$$

N is the term next after W, and consists of the roots

$$\boxed{\overline{n-2p}, \overline{n-2p}, \overline{3p+1}, p}$$

Each of these will traverse the square diagonally, the algebraic sums of

* See Diagrams Nos. 2, 3, and 4.

their roots being equal to the index; the roots of W may also be obtained from A by taking its roots down vertically, making n, n successively $\overline{n-1}$, $\overline{n+1}$, $\overline{n-2}$, $\overline{n+2}$, &c., until the square is reached, in which the index is $2n+2p+1$, and $2n+2p+1$ being the arithmetical sum of all the roots of A; A here becomes a term in a series which moves diagonally, and on being carried up towards the left will give the roots of W. When the indices of the squares through which this vertical series passes equal $2n\pm1$, by making $\overline{p+1}$, p, one positive and the other negative, the term becomes a term in another series which moves diagonally upwards towards the left. This must occur both when the index equals $2n-1$ and when it equals $2n+1$; and as the indices increase downwards uniformly by 2, it follows that $2n-1$ and $2n+1$ will be the indices of contiguous terms of the vertical series, and therefore two contiguous terms will become terms of series moving diagonally upwards to the left; and as these two series are contiguous to each other, their terms found in the first series (that is, the series in the top line) will also be contiguous.

These two terms I have designated as AM and AN. AM comes from the term where the index equals $2n+1$, and AN from the term where the index equals $2n-1$; the roots of AM are

$$\left|\overline{0,\ 2p+1},\ n-\overline{2p+2},\ n,\right|$$

those of AN are

$$\left|\overline{0,\ 2p+1},\ n-2p,\ n,\right|$$

and being arranged thus,

$$\left|\overline{-2p+1},\ 0,\ n-\overline{2p+2},\ n,\right|\ \left|\overline{-2p+1},\ 0,\ \overline{n-2p},\ n,\right|$$

will move diagonally downwards to the left; and as each of these have two roots that differ in one case by $2p$, in the other by $2p+2$, the terms in these series that are parallel to those terms in the first vertical series which have their external indices respectively $2p$ and $2p+2$, the terms of AM and AN will (I say) in these places become terms in series moving vertically, and on being followed up to the series in the top row will be found to give the roots of M and N. The roots of M and N may be obtained by another method as follows:—As the algebraic sum of the roots of A equals its index, therefore A is a term in a diagonal series moving downwards towards the left; and as two of its roots, p, $p+1$, differ by 1, it follows that whenever the value of n has been so altered that $2n\pm1$ equals the index by making $\overline{p+1}$, p, one positive and the other negative, the term becomes a term in another series, which series will move at right angles to the series last mentioned. Now taking the series which moves to the left of A, it is clear that this result will obtain in two places; first, when the altered value of n makes $2n-1$ equal the index, and secondly, when the altered value of n makes $2n+1$ equal the index.

The first of these going up gives the roots of N, the second gives those of M, and these roots are identical with those obtained from AM and AN.

The index of A is $2n-2\overline{p+1}$, therefore when n in moving downwards becomes $\overline{n+1}$, $\overline{n+2}$, $\overline{n+3}$, ...

$$\left|\overline{2n-3p+2},\ \left|\overline{2n-3p+1}\right.\right.,$$

$\overline{p+1}$, p being two of the roots of each term in this series; by adding $p+1$ to the first-mentioned root and p to the other, these two terms will be found to be terms in two horizontal series, of which the first terms are in the first vertical series, and these terms both of them make the diagonal index, and therefore are terms in a diagonal series which, rising towards the right, give the roots of W.

A descends diagonally to the left, and on each side of the line which leads to W changes to cross diagonals which lead to M and N. W leads diagonally to the left to R and S, and where it crosses AT changes and goes up to A. M goes down into R, and then diagonally to where it meets the horizontal series from T; its roots there correspond with the series from T; it returns to T and up to A. In like manner N goes down through S to the line from V, and so to V and up to A. R and S go each of them across to the vertical line from A, and so up to A: every term through which these lines pass has the four roots indicated whose squares would make the term.

The number of terms in the whole square, whose four roots may be expressed in term of p and n, is very considerable; and it may be well now to present some skeleton diagrams of the many ways in which certain members of the square are *invariably* connected.

I propose to exhibit several (to avoid confusion, which would arise from putting all in one diagram); these do not by any means include all the connexions that exist; but whatever may be the value of n or p, a number of the form $(4p+2).n+1$, whether it be $2n+1$, $6n+1$, $10n+1$, &c., and whatever be the value of n, gives the following results. See diagram No. 2. At the $(n-p)$th term there will be A, which descends vertically till the index is $2n-1$, then $2n+1$, and from these rise up in diagonals AN and AM, as already mentioned. A then further descends till the index is equal to $2n+2p+1$, when it rises in a diagonal to W.

Diagram No. 3 shows certain connexions between AM and AN and other terms in the square. AM is always $-\overline{2p+1}$, 0, $n-\overline{2p+2}$, n, AN is always $-\overline{2p+1}$, 0, $\overline{n-2p}$, n.

If the series in which AM is a term be carried down diagonally till 0 becomes $2p+1$ (that is $2p+1$ places), it becomes a term in a diagonal series that intesects it and rises to the top, then goes down to the horizontal series from R, where it becomes a term in that series and passes to where it is below M and rises vertically up to it. AN does the same with respect to the series from S, and rises up to N.

If the term AM descends till $n-\overline{2p+2} = 2p+1$, it rises to the left in another diagonal series and goes on till it crosses M, where it is found that the roots are always the same as those which arise from M, descending by means of its two equal roots. The term AN does the same with respect to N.

If AM descend to the margin at *am*, and one step further into *an*, and AN descends to *an*, they will be found to have the same roots, and they will be

from AM
$-\overline{2p+1}$, 1, $n-\overline{2p+1}$, n
from AN
$-\overline{2p+1}$, -1, $n-\overline{2p+1}$, n

The only difference being that in the one case 1 is positive, in the other it is negative; but whatever be the value of p or n, in this portion or term of the square 1 is *always* one of the roots.

In crossing the two horizontal series from R and S, it will always be found that at the points of intersection the roots of AM correspond with the roots of the series from R, and the roots from AN correspond with those from S.

Diagram No. 4 exhibits the way in which B, C, P, and Q are connected together. The roots of A at the $(n-p)$th square will always be $-(p+1)$, $-p$, n, n, and $p+1, p$ must be one of them odd, the other even; therefore, whether n be odd or even, an odd number will be formed by $-(p+1)$, n, or $-p$, n. The series from A descending diagonally has the roots n, n decreasing, but the negative roots increasing. The differences will continue the same; and when the series arrives under that index which corresponds to the odd number $-(p+1)$ n, or $-p$, n, it becomes a term in a horizontal series which goes to P, two terms of which are always p, $p+1$. W, in descending diagonally, has its index on reaching PB 1. When A has descended so as to reach the even number of the two, $(-(p+1), n, -p, n)$, it rises in a vertical series to Q; and the three series, WR, BP, CQ, always intersect in the same point or small square, H.

The paper then exhibits in a Diagram (No. 5) all the roots which arise from applying the *method* to the odd No. 161, and shows that the roots

of A would be	3,	2,	16,	16
of W ,, ...	3,	2,	6,	16
of M ,, ...	8,	3,	10,	10
of N ,, ...	7,	2,	12,	12
of AM ,, ...	5,	0,	10,	16
of AN ,, ...	5,	0,	12,	16

It shows the roots of the squares marked B, C, P, Q, R, S, and many others; and, finally, it shows that 161 would be composed of the squares

of the following roots, either 3, 12, 2, 2, or 10, 0, 5, 6; but to set forth the roots of the numbers which are in Diagram No. 1, would require a diagram larger than could conveniently be put into a publication of the size of the Proceedings. The roots 10, 0, 5, 6, if arranged thus, -10, 0, 5, 6, have 1 for their sum; but it was proved in the former paper (see p. 410, Trans. Roy. Soc. for 1861) that if the algebraic sum of the roots be 1, then the number is the double $+1$ of a number composed of 3 triangular numbers, $161=80\times2+1$, and 80 is composed of the 3 triangular numbers 55, 15, 10. If therefore any number be doubled, and 1 be added, an odd number will be obtained, to which the same process may be applied as is here applied to 161.

I have stated that the cause of the success of "*the method*" (though it does not at present amount to a demonstration) may be easily shown. It arises, first, from "*the method*" requiring every odd number that is a term in any of the series to be represented by the roots of the square numbers that compose it; and secondly, and more particularly, from every series being connected with at least six others of a different kind which intersect it, each of which is again connected with at least five others, so that when the whole network has been pursued, and the roots which in succession form every term have been recorded, it will be found that many different modes of dividing each term into four squares or less will be discovered, *i. e.* if the numbers be large. I propose to show the manner in which the series are apparently interwoven by an example from each kind.

Let the first term in a horizontal series be 22 | 3, 7, 13, 14 | 21 / 23 with 22 as the index in the margin, and 21 and 23 being the indices of the diagonal series which pass through this square; for, except at the top line, two diagonal series pass through every square. 13 and 14 are the variable roots, which become 12, 15, | 11, 16, | 10, 17, | 9, 18, | &c. | in the successive terms; when in the second term 14 becomes 15 ($15+7=22$), and the roots are 3, 7, 12, 15, a term in the series which would come down from the second square; the roots in that square will therefore be 3, 12, 4, 4; when 15 becomes 19 the roots will be 3, 7, 8, 19, and the roots at the top will be 7, 8, 8, 8; so when 15 becomes 25 at the tenth term the roots are 3, 7, 2, 25; and as -3, $25=22$, another series rises up with roots of the first term, 7, 2, 14, 14.

13 diminishes to 0, and then increases, giving 2 more when it becomes 15 or 19; but the series is also crossed by diagonal series; and when the index in the two rows from the first term $\left\{\begin{matrix}21 & 19 & 17 & 15 & 13, \&c.\\ 23 & 25 & 27 & 29 & 31, \&c.\end{matrix}\right\}$ is 37 (the sum of all the roots), or 31 (the sum of the variable plus the difference of the constant roots), or 17 (the sum of the variable roots minus the sum of the constant roots), or 23 (the sum of the constant roots minus the difference of the variable), a diagonal series arises. Here are no less

than 10 other series by which this is crossed and associated and connected, and the number cannot in any case be less than **6**. A series of the second kind gives rise to other series crossing it in the same manner *mutatis mutandis*, which result is so obvious that it is not necessary further to dwell upon it. A series of the third kind has all the differences of its roots the same in each term. Let $\overset{13}{\boxed{-7,\ 2,\ 4,\ 14}}$ be a term in a diagonal series at the top row of the system.

Thus:

	11	13	15
0	8, 13, 2, 2	9 2 10 −7, 2, 4, 14	6, 15, 4, 4
	9		17
2	9 2 10 −8, 1, 3, 13		9 2 10 −6, 3, 5, 15

When it reaches to the left and to the right, the second place, as $3-1=2$ and $5-3=2$, it furnishes two vertical series; at the tenth row it furnishes two more; at the twelfth row two more. When it gets into the column whose index is 11 and then 9, before it reaches the margin or outer edge, it furnishes two horizontal series; and after it has passed the margin at 9 and 11 it furnishes two more, and at 21 it furnishes another. Here are six new vertical and five new horizontal series; besides which it furnishes at least two other diagonal series which cross it.

Having stated the properties which belong to the square, if the first odd number in it be of the form $(4p+2).n+1$, and that whether it be $2n+1$, $6n+1$, $10n+1$, &c., or whatever be the value of n, certain squares may be found which I distinguish as A, B, C, H, P, Q, W, M, N, AM, AN, R, S, T, V, &c., which are connected together by a community of roots where the series cross each other in a manner that is invariable.

The Diagram No. 3 is another example of the manner in which certain of the terms in the different series communicate with each other, by the roots being common to both, at the point where they cross. AM passes diagonally to *am*, down to *an*, and up to AN. AN, in like manner, goes down to *an* and up to *am* and AM. If the first term of the square be an odd number of the form $(4p+2)\ n+1$, the roots from AM are in an $-(2p+1)$, 1, $(n-2p+1)$, n. The roots from AN are $-(2p+1)$, -1, $(n-2p+1)$, n. The indices of the diagonal series are $\begin{pmatrix}2n-(4p+3)\\2n-(4p+1)\end{pmatrix}$, and the algebraic sum of the roots is the one or the other, according as 1 is + or −; but AM also passes to the series from R, and AN to the series from S, and go to R and S, and thus go up to W. AM also reaches the vertical line from M, and passes up to M, as AN does to N. Lastly, AM goes to N thus, and AN to W in a similar manner. Whatever be the

Diagram N° 1

$10n + 1$

$n = 16$

$p = 2$

	1	3	5	7	9	11	13	15	17	19
0	161	165	173	185	201	221	245	273	305	341
2	163	167	175	187	203	223	247	275	307	343
4	169	173	181	193	209	229	253	281	313	349
6	179	183	191	203	219	239	263	291	323	359
8	193	197	205	217	233	253	277	305	337	373
10	211	215	223	235	251	271	295	323	355	391
12	233	237	245	257	273	293	317	345	377	413
14	259	263	271	283	299	319	343	371	403	439
16	289	293	301	313	329	349	373	401	433	469
18	323	327	335	347	363	383	407	435	467	503
20	361	365	373	385	401	421	445	473	505	541

Diagram Nº 2.

Proc. Roy. Soc. Vol XV. Pl. IV.

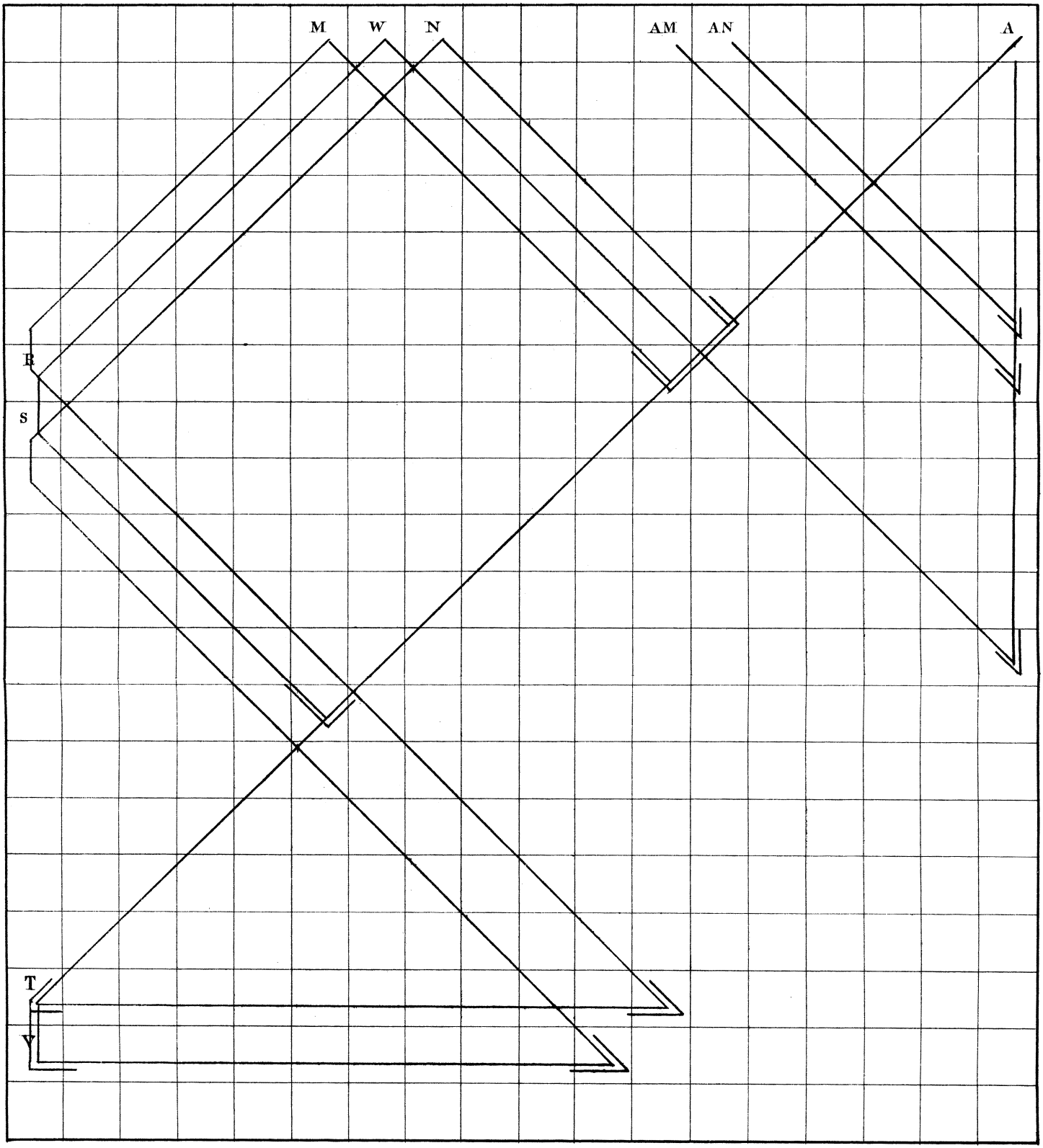


Diagram N° 3

Proc. Roy. Soc. Vol XV. Pl. V.

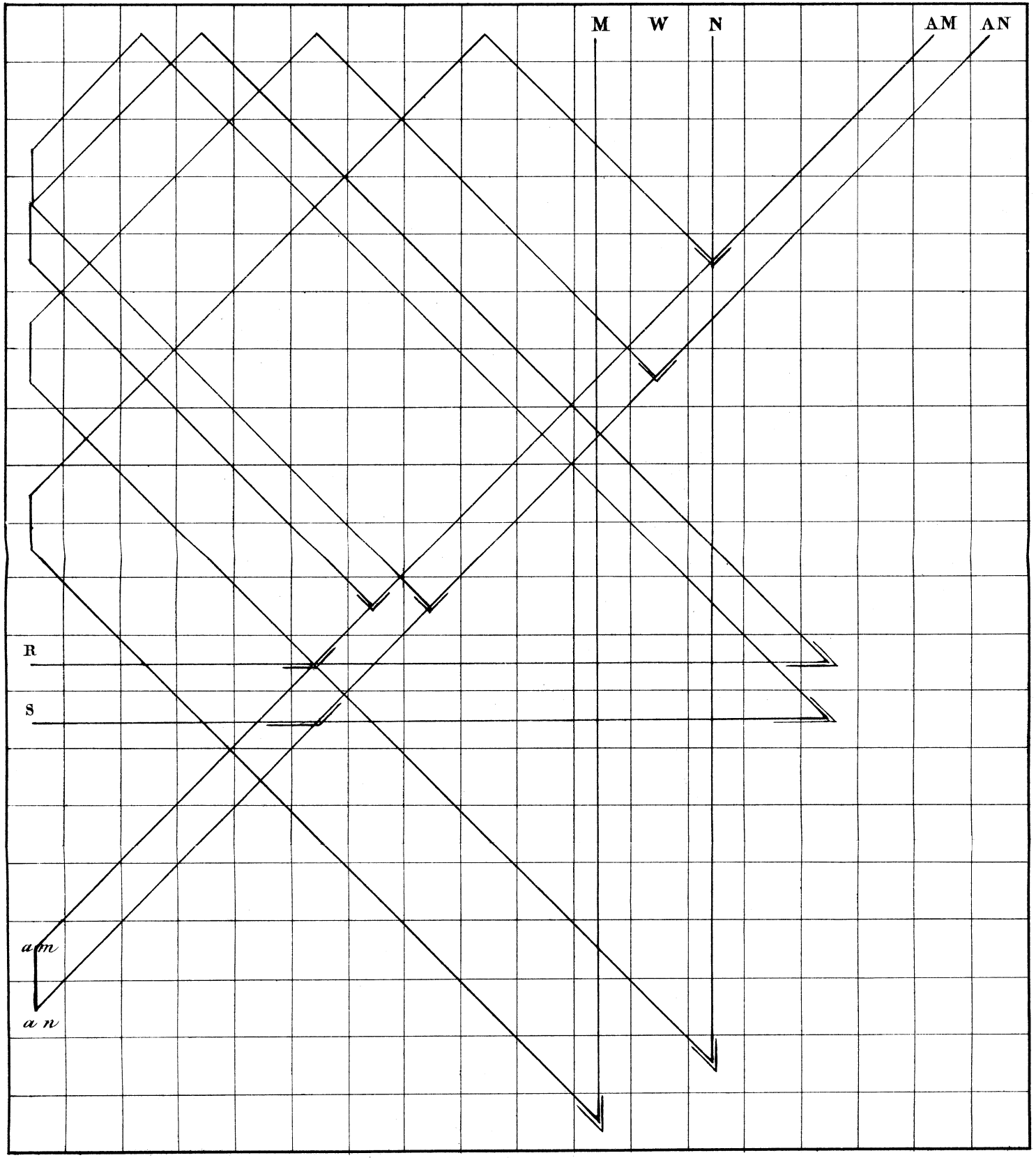
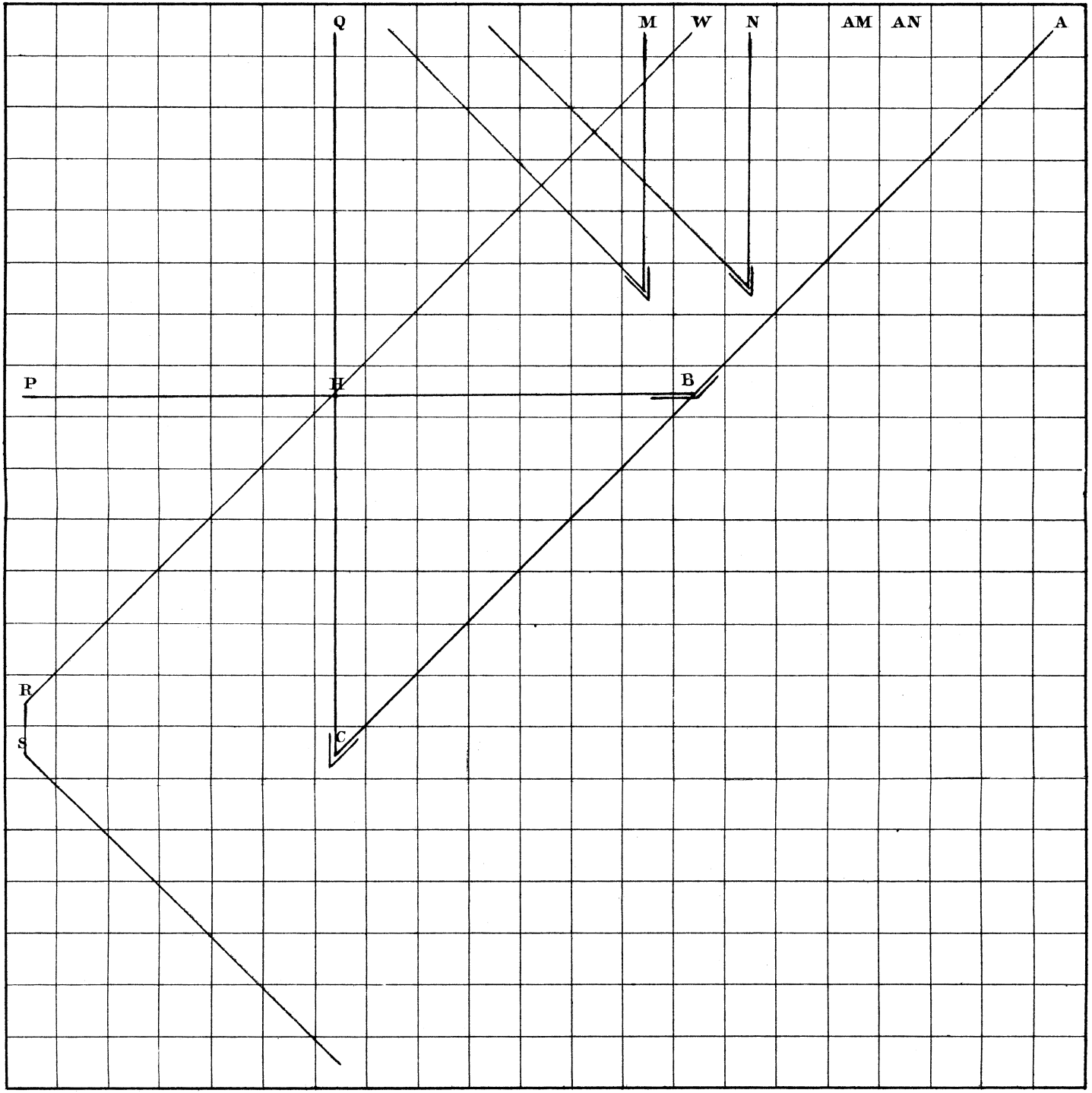


Diagram N° 4.

Proc. Roy. Soc. Vol XV. Pl. VI.



values of p and n, these connexions occur, but the form of them varies with the values of p and n.

I have pointed out some of the results of having as the first term in the square an odd number of the form $(4p+2)$, $n+1$; the forms will be $2n+1$, $6n+1$, $10n+1$, $14n+1$, &c., in which p may be 0, 1, 2, 3, &c. But there is *another* form which gives similar results, viz. $4pn+(2p+1)$, which is in many respects the "*converse*" of the other; the forms will be $4n+3$, $8n+5$, $12n+7$, $16n+9$, &c. These are the *first terms* to which this system gives rise. The $(n-\overline{p-1})$th term is always $\boxed{p, p, n, n+1}$. In the former system n produced the equal roots and p the unequal; *here*, p produces the equal and n the unequal roots, and I call the $(n-\overline{p-1})$th term A as in the other. This system has also W and M and N on each side of it, and other squares similar to the other, but the roots of which they are composed are differently formed. M and N come from below. W, instead of being

$\boxed{-\overline{p+1}, -p, n-4p+2, n,}$ will be $\boxed{-3p, p, (n-2p), (n+1-2p),}$

and other terms are similarly altered, but the general result is the same. A specimen of a square of this form is given in Diagram No. 8, but which cannot be reduced to the size of the Proceedings, where $p=3$ and $n=16$; the first number is (199), and the square is completed so far as to show that the roots of the 4 squares whose sum is (199) $=$

$$\boxed{\begin{array}{rrrr} 5, & 13, & 1, & 2 \\ -9, & -3, & 3, & 10 \\ 1, & 10, & 7, & 7 \end{array}}$$

but neither in this Diagram nor in Diagram No. 5 is the whole square completed (to avoid confusion); but if the series be traced in succession, the entire Diagram 6 would be filled up, and every term would disclose the roots whose squares compose it. In this manner every odd number in all the series is divided into the squares that compose it (not exceeding 4), the squares being indicated by their roots.

The two systems of $(4p+2)\times n+1$ and $4p.n+2p+1$ include every possible odd number; $4n+3$ includes every alternate odd number from 3; $8n+5$ every fourth number from 5, and so on; $2n+1$ includes every odd number; $6n+1$ every third odd number. Many odd numbers belong to both systems, and to more than one in each. 151 is an example; it is either $10n+1$ $(n=15)$, or it is $12n+7$ $(n=12)$. The paper contains a Diagram (No. 9) exhibiting the odd number 151 as belonging to both systems; but the Diagram cannot be reduced. The roots of the squares that compose 151 are (10. 1. 5. 5), or (3. 9. 5. 6).

Diagram No. 6.

$2n-\overline{6p+5}$	$2n-\overline{6p+3}$	$2n-\overline{6p+1}$		$2n-\overline{4p+3}$	$2n-\overline{4p+1}$		$2n-\overline{2p+1}$
M	W	N		AM	AN		A
$-\overline{3p+2}$,	$-\overline{p+1}$,	$-3p+1$,		$-2p+1$, 0,	$-\overline{2p+1}$, 0,		$-\overline{p+1}$,
$p+1$,	$-p$,	p, $n-2p$,	an interval of	$n-\overline{2p+2}$, n,	$n-2p$, n,	an interval of	$-p$, n, n
$n-\overline{2p+2}$,	$n-\overline{4p+2}$, n,	$n-2p$,	$\overline{p-2}$ squares.			$\overline{p-1}$ squares.	
$n-\overline{2p+2}$,							

This Diagram (No. 6) is introduced to show in terms of n and p what roots A, AN, AM, N, W, and M contain, and the intervals between them. Each of these in passing downwards to the left is crossed by other series, with which they amalgamate; N may be derived from AN, M from AM: N, W, and M each furnish two others, and these again each two more. When n is small compared with the coefficient $(4p+2)$, W may be on the right of A; for although the series begins at A, and according to the law of the series reaches the 1st square, the same law enables it to continue, with terms whose indices become negative.

Diagram No. 7.

There is an interval of $n-\overline{3p+3}$ squares from the 1st term.

*$2n-\overline{6p+4}$ R	$2n-\overline{6p+5}$ $-\overline{3p+2}$, p $n-\overline{2p+2}$, $n-\overline{2p+1}$
$2n-\overline{6p+4}$ S	$2n-\overline{6p+1}$ $-\overline{3p+1}$, $p+1$ $n-\overline{2p+1}$, $n-2p$

There is here an interval of $(p-1)$ squares from S to *an*.

$2n-\overline{4p+2}$ *an*	$2n-\overline{4p+3}$ $2n-\overline{4p+1}$ $-\overline{2p+1}$, 1 $n-\overline{2p+1}$, n

There is here an interval of $(p-1)$ squares from *an* to T.

$2n-\overline{2p+2}$ T	$2n-\overline{2p+3}$ $p+1$, $p+1$ $\overline{n-1}$, n
$2n-2p$ V	$2n-\overline{2p-1}$ p, p, n, $n+1$

This Diagram (No. 7) shows in terms of n and p the roots of R, S, *an* (which comes down diagonally from AN), T, and V, and the number of squares between them; the relative positions of these terms depend entirely on p, and are always the same for the same value of p.

Terms similar to these increase indefinitely as n increases. The value of certain roots is independent of n, and therefore is the same for every value of n.

* [The numbers above the letters are the indices of the vertical series.]

Diagram № 1

10 n + 1

n = 16

p = 2

	1	3	5	7	9	11	13	15	17	19
0	161	165	173	185	201	221	245	273	305	341
2	163	167	175	187	203	223	247	275	307	343
4	169	173	181	193	209	229	253	281	313	349
6	179	183	191	203	219	239	263	291	323	359
8	193	197	205	217	233	253	277	305	337	373
10	211	215	223	235	251	271	295	323	355	391
12	233	237	245	257	273	293	317	345	377	413
14	259	263	271	283	299	319	343	371	403	439
16	289	293	301	313	329	349	373	401	433	469
18	323	327	335	347	363	383	407	435	467	503
20	361	365	373	385	401	421	445	473	505	541

Diagram № 2.

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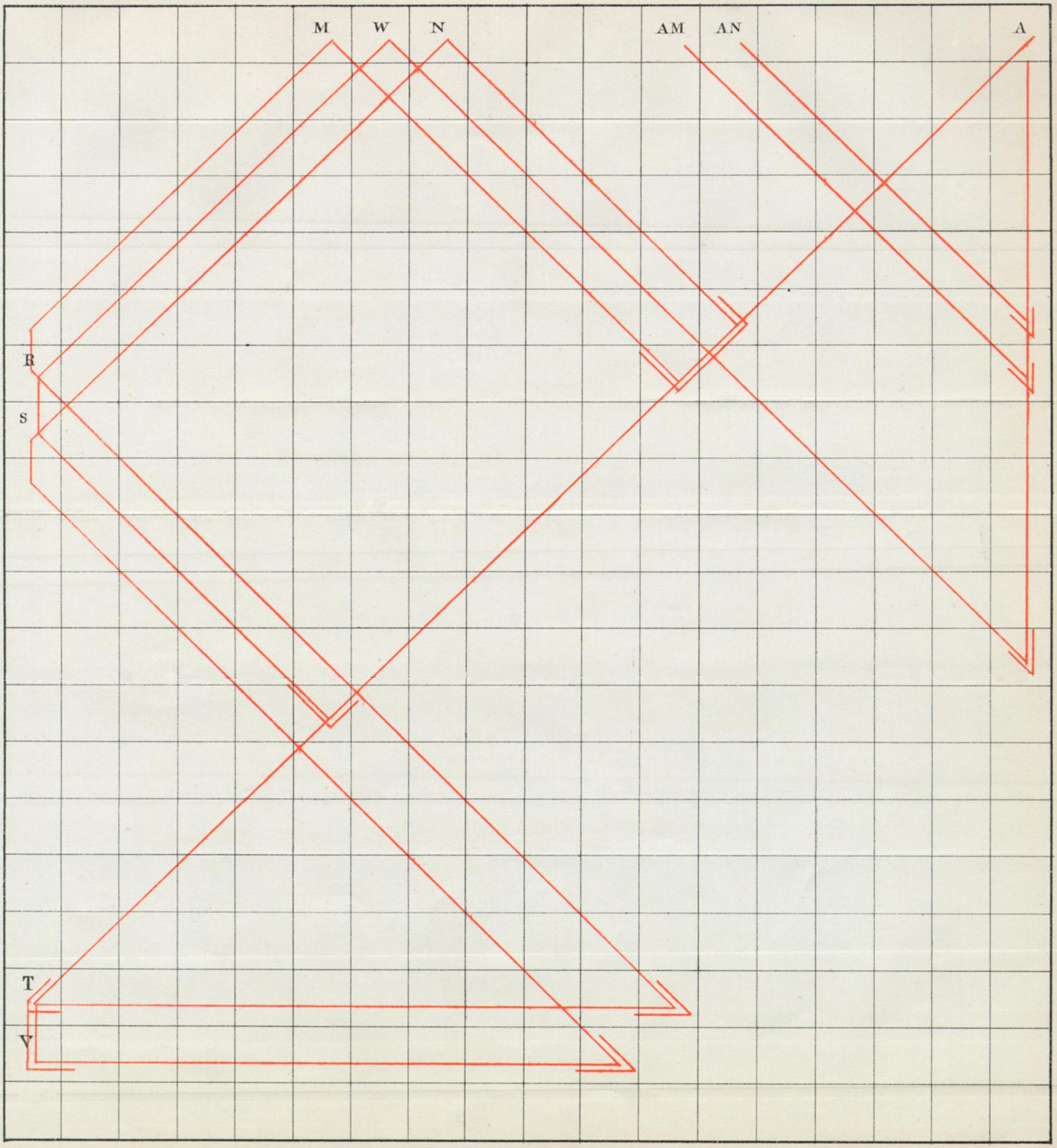

Diagram Nº 3

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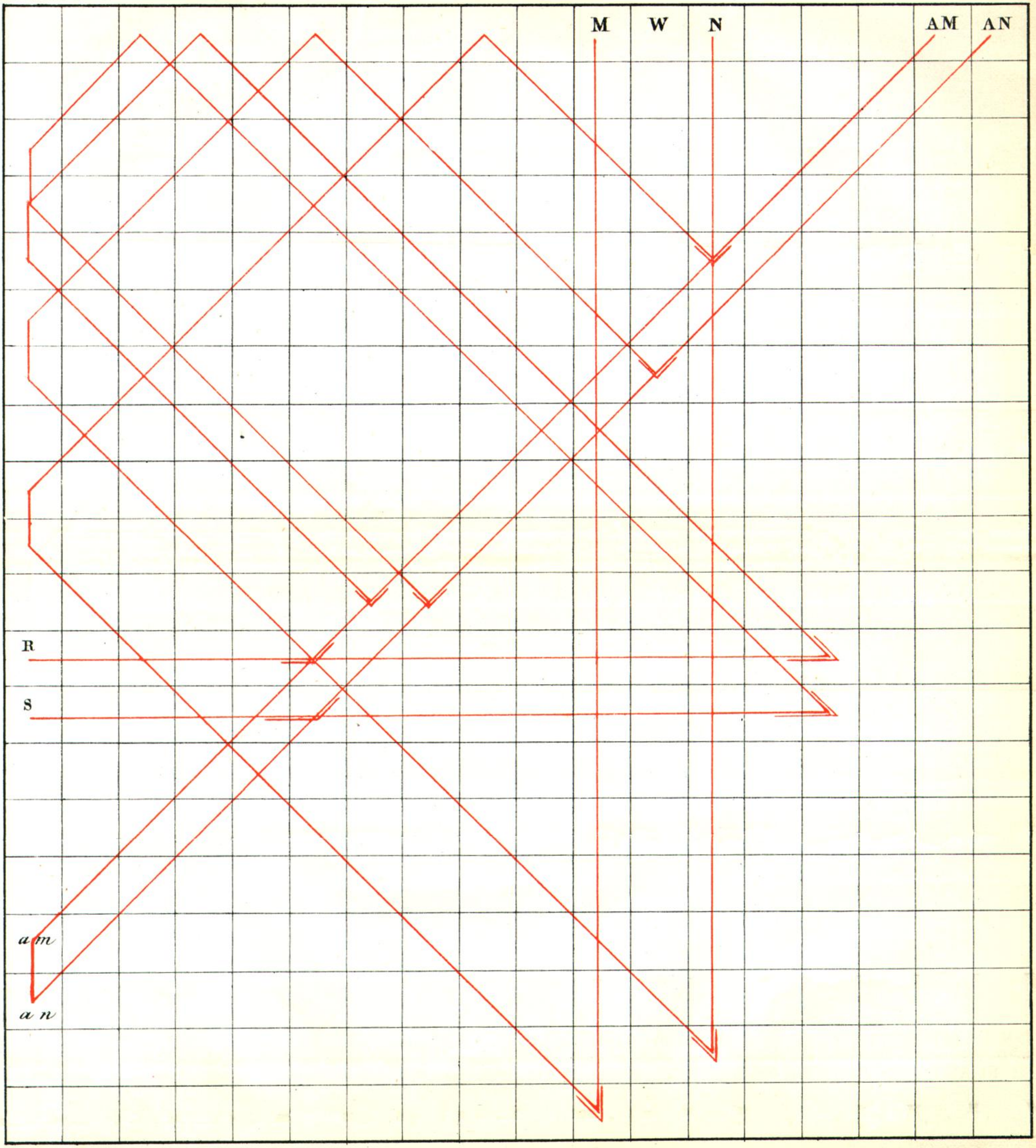

Diagram N°. 4.